GENERAL RADIO COMPANY

engineering department

INSTRUMENT NOTES

IN-103 1967

USEFUL FORMULAS, TABLES, AND CURVES FOR RANDOM NOISE

This collection was originally prepared for use by General Radio people who work with random noise occasionally, but do not have all the formulas reliably memorized. It has proved so useful that we are making it available to others who may be interested. Suggestions for additions to this collection will be welcomed for possible inclusion in subsequent printings.

SPECTRUM DENSITY

Spectrum density is the power per unit bandwidth and in general is a function of frequency. If spectrum density is constant between frequencies f_1 and f_2:

$$W_n = \frac{W_t}{\Delta f_c} \text{ watts/cps}$$

Where: W_n = spectrum density between frequencies f_1 and f_2

W_t = total noise power in frequency band between f_1 and f_2, watts

$\Delta f_c = f_2 - f_1$ = bandwidth over which spectrum density is uniform, cps.

Noise having uniform spectrum density over a specified frequency range is said to be "white" in that range. Noise whose spectral density varies inversely with frequency is said to be "pink". "White" noise has constant energy per unit bandwidth; "pink" noise has constant energy per octave.

THERMAL NOISE GENERATED IN A RESISTANCE

$e_{nr} = \sqrt{4kTR\Delta f_c}$ volts, rms, and $i_{nr} = \sqrt{\dfrac{4kT\Delta f_c}{R}}$ amperes, rms.

Where: e_{nr} = open-circuit thermal noise voltage

i_{nr} = short-circuit thermal noise current

k = Boltzmann's constant = 1.38×10^{-23} joule/°K

R = resistance, ohms

T = absolute temperature of resistor, °K

Δf_c = effective noise bandwidth, cps

Δf_{Mc} = effective noise bandwidth, Mc (used in formula below).

Available thermal-noise-power spectrum density (W_a) from a resistance R into a matching load R_L ($R_L = R$), at frequencies where effects of residual inductance and capacitance are negligible is independent of the magnitude of the resistors: W_a = kT watts/cps. *Net* power-density transfer from R to R_L, considering noise power generated by *both* resistances, equals kΔT, where ΔT is the difference in absolute temperatures of the resistors.

At a standard temperature of 290°K (=17°C = 62.6°F):

$e_{nr} = 1.27 \times 10^{-10} \sqrt{R\Delta f_c}$ volts rms.

For R = 50 ohms and Δf = 1 Mc, e_{nr} = 0.90 microvolt.

For R = 1 megohm and Δf = 50 cps, e_{nr} = 0.90 microvolt.

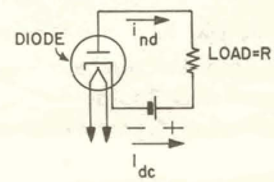
For other conditions:

$$e_{nr} = 0.90\sqrt{\left(\frac{R}{50\Omega}\right)\left(\frac{T}{290\,°\text{K}}\right)\left(\frac{\Delta f_{Mc}}{1\text{ Mc}}\right)} \text{ microvolts, rms.}$$

Near room temperature, a temperature increase of 10°F produces an increase in noise voltage of about 1%.

SHOT NOISE FROM TEMPERATURE-LIMITED NOISE DIODE

Low-frequency equivalent circuit.



$i_{nd} = \sqrt{2eI_{dc}\Delta f_c}$ amperes, rms

Where: i_{nd} = shot-noise current, amperes

e = charge of electron = 1.602×10^{-19} coulomb

I_{dc} = diode dc current, amperes

Δf_c = effective noise bandwidth, cps

Δf_{Mc} = effective noise bandwidth, Mc (used in formula below).

Conditions:

1. Plate-cathode voltage must be high enough so that current does not become limited by space charge.
2. Frequency must be low enough to avoid transit-time effects and high enough to avoid flicker-noise effects.

Spectrum density, due to shot noise only, in R equals

$$2eI_{dc}R = 3.2 \times 10^{-19}\, I_{dc}R \text{ watts/cps.}$$

Spectrum density, due to shot noise only, available from shunted diode to external load of resistance equal to internal shunting resistance, R, equals

$$\frac{e}{2} I_{dc} R = 8.0 \times 10^{-20} I_{dc} R \text{ watts/cps.}$$

NOISE FACTOR (F)

For a linear two-port device with a specified input termination:

$$F = \frac{\text{Available noise power output arising from all sources}}{\text{Available noise power output due to thermal noise of source impedance alone*}}$$

$$F = \frac{\text{Noise power arising from all sources referred to input of two-port device}}{\text{Noise power at input of two-port device due to thermal noise of source impedance alone*}}$$

If a coherent signal is involved:

$$F = \frac{\text{Signal-to-noise power ratio of source*}}{\text{Signal-to-noise power ratio at output of two-port-device}}$$

For a perfectly noiseless two-port device, $F = 1$.

Spot Noise Factor, $F(f)$: In the above definitions, noise power is per unit bandwidth at a specified frequency f, measured in a bandwidth sufficiently small that the noise spectrum density is constant in that bandwidth.

Average Noise Factor, $\bar{F}$: In the above definitions, noise power is the average value over a wide bandwidth. If spot noise factor is constant over a specified bandwidth, then it is equal to the average noise factor in that bandwidth.

Excess Noise Factor: The amount by which the noise factor exceeds unity. For example, if $F = 1.2$, excess noise factor = 0.2.

NOISE FIGURE, NF

The term "noise figure" is often used interchangeably with the term "noise factor", but is also sometimes defined as noise factor expressed in decibels. $NF = 10 \log_{10} F$. For a perfectly noiseless two-port device, NF = 0 db. If $F = 2$, NF = 3 db.

EQUIVALENT NOISE TEMPERATURE (T_n) FOR A SINGLE-PORT (TWO-TERMINAL) DEVICE

The temperature at which its resistance would produce the observed total available noise:

$$T_n = \frac{W_n}{k} \text{ degrees Kelvin}$$

Where: W_n = total available noise power per unit bandwidth produced by all sources within device, watts/cps

k = Boltzmann's constant = 1.38×10^{-23} joule/°K.

EFFECTIVE NOISE TEMPERATURE (T_n) FOR A TWO-PORT (FOUR-TERMINAL) DEVICE

The temperature at which the source resistance would generate the same total output noise power as is actually generated by all noise sources *within* the device.

$$T_n = 290\,(F - 1) \text{ degrees Kelvin}$$

Where: F = noise factor.

NOISE RATIO (r_n)

For a single-port device, the ratio of the effective noise temperature to the actual temperature:

$$r_n = \frac{T_n}{T_s}$$

Where: T_n = effective noise temperature, °K

T_s = actual temperature, °K.

NOISE-FACTOR MEASUREMENT WITH NOISE DIODE

Noise diode connected to input, and a detector connected to output, of two-port device. Diode d-c current first set to zero, and output noise power into detector noted; then diode d-c current increased until output noise power is twice initial value.

$$F = 20\, I_{dc} R_s$$

Where: F = standard spot noise factor of two-port device, with input terminated in resistance R_s at standard temperature of 290°K, at any frequency within the effective noise bandwidth of device and detector combined, assuming device noise is white over that bandwidth. If not, then F is the standard average noise factor in the effective bandwidth

I_{dc} = noise diode d-c current in amperes required to double noise power

R_s = effective source resistance in ohms of noise diode circuit as connected to input of two-port device.

This method is accurate for noise factors greater than about 2 (noise figures greater than about 3 db). Below these values, the source-resistance thermal noise, which is always present, is larger than the two-port-device noise of interest, and the partial masking reduces measurement accuracy.

NOISE-FACTOR MEASUREMENT WITH CW SIGNAL GENERATOR

CW signal generator connected to input, and rms detector connected to output, of two-port device. Signal generator output first set to zero, and output noise power into detector noted; then signal generator voltage increased until rms power into detector is twice initial value.

$$\bar{F} = \frac{62.5\, E_s^2}{R_s \Delta f_{Mc}}$$

Where: $\bar{F}$ = standard average noise factor of two-port device, with input terminated in R_s, within the effective noise bandwidth of device and detector combined

R_s = effective source resistance, in ohms, of cw signal generator

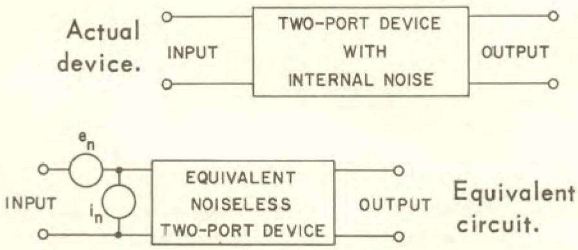
Δf_{Mc} = effective noise bandwidth, in Mc, of two-port device and detector combined

*Operated at a standard temperature of 290°K = 17°C = 62.6°F.

E_s = equivalent open-circuit rms output voltage, in μv, of cw signal generator required to double output power into detector.

This method is most useful for the measurement of very large noise factors, where the limited output power of available noise sources would restrict their use.

USE OF EQUIVALENT CONSTANT-VOLTAGE AND CONSTANT-CURRENT GENERATORS TO REPRESENT TWO-PORT-DEVICE NOISE AND TO MEASURE NOISE FACTOR



Conditions:

Device must be linear, and stable when input is either short-circuited or open-circuited.

e_n = equivalent input rms noise voltage, in volts, that would produce the same noise output as is produced by the device with input terminals short-circuited. Measured by connecting a low-impedance* known voltage source** to device input and setting its amplitude to produce a known large ratio of increase (say 100:1) in device total output power. Use of a narrow-band detector is assumed.

i_n = equivalent input rms noise current, in amperes, that would produce the same noise output as is produced by the device with input terminals open-circuited. Measured by connecting a high-impedance* known current source** to device input and setting its amplitude to produce a known large ratio of increase (say 100:1) in device total output power. Use of a narrow-band detector is assumed.

* with respect to device input impedance.

**either sine wave of known amplitude or broadband noise with known spectrum density.

Using the sine-wave method: Since the values obtained for e_n and i_n are integrated values over the effective noise bandwidth, Δf_c in cycles, of the device and detector combined, this bandwidth must be known in order to calculate average power spectrum density and noise factor.

Using the noise-source method: The values obtained are $\frac{e_n}{\sqrt{\Delta f_c}}$ and $\frac{i_n}{\sqrt{\Delta f_c}}$, because they are measured in terms of known spectrum density of the noise generator, so that bandwidth need not be known for the calculation of noise factor.

Optimum Source Resistance: At low frequencies where source impedance is essentially resistive:

Optimum source resistance for minimum noise factor $\equiv R_o = \left|\frac{e_n}{i_n}\right| = \left|\frac{e_n/\sqrt{\Delta f_c}}{i_n/\sqrt{\Delta f_c}}\right|$.

Excess noise factor for optimum source resistance:

$$\left(F_o - 1\right) = \left(1 + \rho\right)\left(\frac{10^{21}}{8}\right)\left(\frac{e_n i_n}{\Delta f_c}\right) = \left(1 + \rho\right)\left(\frac{\mu v \text{ x } pa}{8 \text{ x kc}}\right)$$

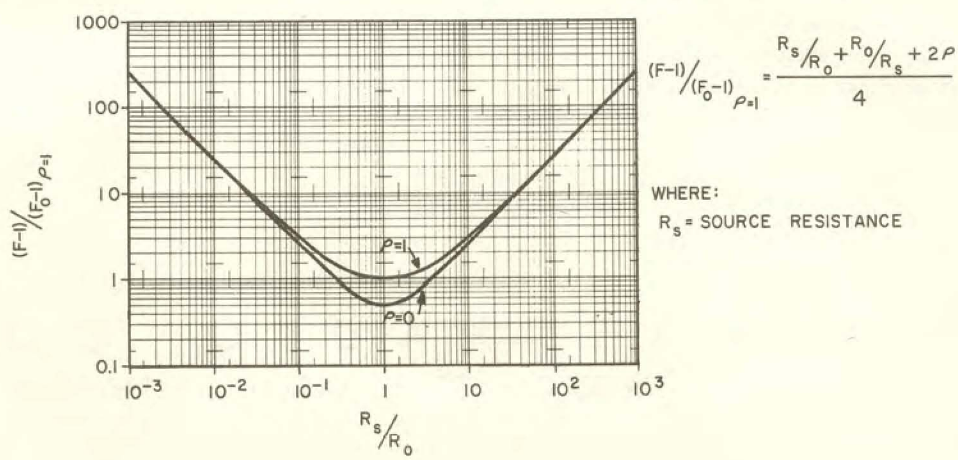
Where ρ = correlation coefficient between e_n and i_n. Theoretically, ρ can be complex, and $0 \leqq |\rho| \leqq 1$.

For a simple device, within which most of the noise arises from a single source:

$$\rho \cong 1; \quad \text{and} \quad \left(F_o - 1\right) \cong \frac{\mu v \text{ x } pa}{4 \text{ x kc}}.$$

This method is particularly useful for measuring low-noise devices, since the device noise is measured alone, without the masking presence of appreciable thermal noise from a source resistance. Also, this method provides simply the value of the optimum source resistance for lowest noise factor and the noise factor for any other value of source resistance.

(Reference: Sanderson and Fulks, "A Simplified Noise Theory and Its Application to the Design of Low-Noise Amplifiers," IRE Transactions on Audio, July-August, 1961).



Universal curve for excess noise factor *vs* source resistance.

EFFECTIVE NOISE BANDWIDTH ($\triangle f$)

The effective noise bandwidth of a device is equal to the bandwidth of an ideal rectangular-bandpass filter having a passband gain equal (usually) to the maximum device gain and which would transmit the same total power from a white-noise source as would the device.

In general:

$$\triangle f = \frac{1}{G_{max}} \int_0^\infty G(f)\, df, \text{ cycles}$$

Where: $G(f)$ = absolute magnitude of device power gain as a function of frequency

$\frac{1}{H^2_{max}} \int H^2(f) df$ G_{max} = maximum value of $G(f)$.

EFFECTIVE NOISE BANDWIDTH OF MAXIMALLY FLAT BAND-PASS FILTERS

Equivalent Number of Tuned Circuits	(Effective Noise Bandwidth) ÷ (3-db Bandwidth)	Attenuation at Limits of Effective Noise Bandwidth
1*	$\frac{\pi}{2} = 1.571$	5.40 db
2**	$\frac{\pi}{2\sqrt{2}} = 1.111$	4.02 db
3	$\frac{\pi}{3} = 1.047$	3.65 db
5	1.017	3.39 db
20	1.001	3.10 db

*Single tuned circuit.

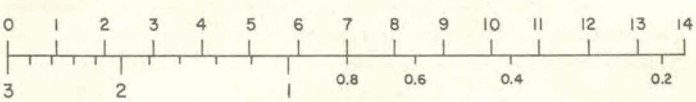
**Pair of transitionally coupled tuned circuits.

(Reference: Valley and Wallman, Vacuum-Tube Amplifiers, McGraw-Hill, 1948, p 169)

ADDING NOISE LEVELS IN DECIBELS

If the levels of two uncorrelated noises, A and B (B higher than A), are measured separately and expressed in decibels with respect to a common reference level, the resultant level in decibels of their combination may be found from the following line chart:

DIFFERENCE IN DECIBELS BETWEEN TWO LEVELS BEING ADDED (B-A)



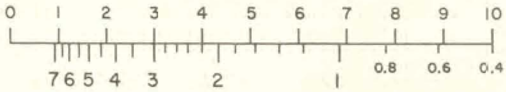
INCREMENT IN DECIBELS TO BE ADDED TO HIGHER LEVEL (B) TO OBTAIN RESULTANT LEVEL OF COMBINATION (A + B)

Example: Noise signal A = 85 db. Noise signal B = 92 db. Difference = B - A = 92 db - 85 db = 7 db. From line chart: Increment corresponding to a 7-db difference is 0.8 db. Therefore, resultant level of A + B = 92 db + 0.8 db = 92.8 db.

SUBTRACTING NOISE LEVELS IN DECIBELS

If one noise level (A) is to be measured in the presence of an uncorrelated background noise level (B) which cannot be avoided, it is necessary to measure their combination (A + B) and to measure separately and subtract the background (B). If all levels are expressed in decibels with respect to a common reference, the difference may be found from the following line chart:

DIFFERENCE IN DECIBELS BETWEEN COMBINATION OF NOISE AND BACKGROUND (A + B) AND BACKGROUND ALONE (B)



DECIBELS TO BE SUBTRACTED FROM COMBINATION (A + B) TO OBTAIN VALUE OF A ALONE

Example: Combination of noise and background (A and B) = 60 db.
Background alone (B) = 56 db.
Difference = (A and B) - (B) = 60 db - 56 db = 4 db.
From line chart: Amount to be subtracted corresponding to a 4-db difference is 2.2 db. Therefore, level A alone is (60 db - 2.2 db) = 57.8 db.

(Reference: Peterson and Gross, Handbook of Noise Measurement, General Radio Company)

RMS-TO-AVERAGE* RATIOS

For Gaussian noise:

$$\frac{\text{RMS value}}{\text{average* value}} = \sqrt{\frac{\pi}{2}} = 1.25, \text{ or } 1.96 \text{ db.}$$

For sine wave:

$$\frac{\text{RMS value}}{\text{average* value}} = \frac{\pi}{2\sqrt{2}} = 1.11, \text{ or } 0.91 \text{ db.}$$

RESPONSE OF VOLTMETERS TO NOISE 1.05db .885

Average* Meter: Random noise with same rms value as a sine wave has an average* value 1.05 db less than the average* value of the sine wave. (1.96 db - 0.91 db = 1.05 db) An average*-responding meter (calibrated to read rms value of a sine wave) will therefore read 1.05 db too low compared to the rms value of a random-noise input.

Average* meter with both sine wave and noise applied: An average*-responding meter with equal rms levels of a random-noise signal and a sine-wave signal applied reads 2.18 db higher than it would for the random noise alone, or 1.13 db higher than it would for the sine wave alone (2.18 db - 1.13 db = 1.05 db).

Peak Meter: The response of a "peak-responding" diode voltmeter to noise depends upon the charge-to-discharge resistance ratio of the diode rectifying circuit. This in turn depends on the diode E-I characteristic, which is a function of voltage level, and on the R and C circuit constants. External resistance in series with the source is sometimes used to average out the noise fluctuations better, at the expense of reducing the upper frequency limit of the meter. For a given model voltmeter, a curve can be drawn as a function of meter indication, and for a given external series resistance, giving the ratio of

*Full-wave-rectified average.

rms noise voltage to indicated voltage. By the use of such a curve, which can be obtained at low frequencies by comparison with an rms or average meter, the peak meter can be useful for noise measurements over a very wide frequency range and, when used without an external series resistance, can yield information about instantaneous peak voltages that cannot be obtained with other types of meters.

(Reference: Peterson, "Response of Peak Voltmeters to Random Noise," General Radio Experimenter, December, 1956)

AMPLITUDE DISTRIBUTION OF GAUSSIAN NOISE

The amplitude density, p(v), when multiplied by a voltage increment dv, gives the probability that at any given instant the noise voltage lies between v and v + dv. The amplitude distribution P(v) is the probability that at any given instant the noise voltage lies below the value v. For Gaussian noise, having no dc component,

$$p(v) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{v^2}{2\sigma^2}\right)}$$

and

$$P(v) = \int_{-\infty}^{x} p(v)\,dv = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{v}{\sigma\sqrt{2}}\right)\right]$$

where σ is the root-mean-square amplitude of the noise voltage. Values of these functions:

v	p(v)	P(v)
-5σ	$0.000,001,487/\sigma$	0.000,000,287
-4σ	$0.000,133,8/\sigma$	0.000,031,67
-3σ	$0.004,432/\sigma$	0.001,350
-2σ	$0.053,99/\sigma$	0.022,75
-1σ	$0.241,97/\sigma$	0.158,65
0	$0.398,94/\sigma$	0.500,00
1σ	$0.241,97/\sigma$	0.841,34
2σ	$0.053,99/\sigma$	0.977,25
3σ	$0.004,432/\sigma$	0.998,650
4σ	$0.000,133,8/\sigma$	0.999,968,33
5σ	$0.000,001,487/\sigma$	0.999,999,713

FLUCTUATION OF RECTIFIED NOISE *vs* BANDWIDTH AND AVERAGING TIME

The averaging time required in a smoothing filter to reduce the fluctuations of a rectified noise signal to a desired level for a given input bandwidth is, for a full-wave linear detector,

$$RC \cong \frac{1250}{\Delta f_c\, \sigma^2}$$

where RC = the averaging time constant of the smoothing filter in seconds,

Δf_c = the bandwidth of the noise, in cycles, and

σ = the standard deviation of the amplitude fluctuations at the output, expressed in percent of the dc output.

For a square-law detector,

$$RC \cong \frac{1250}{\Delta f_c\, \sigma^2}\,.$$

(Reference: van der Ziel, A., Noise, Prentice-Hall, Inc., New York, 1954.)

EXPECTED NUMBER OF ZERO-AXIS CROSSINGS

For "white", random noise with upper and lower frequency limits of f_2 and f_1, respectively, the expected number of zero crossings per second of *either* positive-going or negative-going sense is:

$$N = \sqrt{\frac{f_2^3 - f_1^3}{3(f_2 - f_1)}} = \sqrt{\frac{1}{3}(f_2^2 + f_1 f_2 + f_1^2)}\,.$$

If $f_1 = 0$ (low-pass filter), $N = 0.577\, f_2$.

For narrow-band noise ($f_1 \cong f_2$), $N \cong \dfrac{f_1 + f_2}{2}$.

The number of zero crossings per second in *both* senses is double the values of N given above.

EXPECTED NUMBER OF MAXIMA

For "white", random noise with upper and lower frequency limits of f_2 and f_1, respectively, the expected number of maxima per second is:

$$M = \sqrt{\frac{3\,(f_2^5 - f_1^5)}{5\,(f_2^3 - f_1^3)}}\,.$$

If $f_1 = 0$ (low-pass filter), $M = 0.775\, f_2$.

For narrow-band noise ($f_1 \cong f_2$), $M \cong \dfrac{f_1 + f_2}{2}$.

The number of maxima per second of the absolute value is double the values of M given above.

(Reference: Rice, S. O., "Mathematical Analysis of Random Noise," Bell System Technical Journal, July, 1944)

"REGULARIZING" ACTION OF SCALING ON PULSES ARRIVING AT RANDOM INTERVALS

The extreme variations which occur in pulse-to-pulse spacing when pulses arrive at random intervals are greatly reduced by a scaling, or pulse-division, process. This fact relaxes the speed requirement on a following circuit, for a specified allowable counting loss by that circuit, by a factor considerably greater than the scaling ratio.

For pulses arriving at random intervals (Poisson distribution of time intervals between pulses), see table on next page:

(References: Rainwater and Wu, "Applications of Probability Theory to Nuclear Particle Detection," Nucleonics, October, 1947, pp 60-69.

Pryor and Klein, "Statistical Design Basis for Fast Scaling Systems," Nuclear Instruments and Methods, January, 1959, pp 1-4)

Scaling Ratio (Number of input pulses to produce one output pulse)	Most Probable Interval ÷ Average Interval	Probability Density at Most Probable Interval	Probability Density at Average Interval	Standard Deviation ÷ Average Interval	Dead Time of Following Scaler ÷ Mean Input Interval		
					For 0.1% Counting Loss	For 1.0% Counting Loss	For 10.0% Counting Loss
1	0	1	0.368	1	0.001	0.010	0.11
2	0.50	0.736	0.541	0.707	0.032	0.10	0.34
10	0.90	1.32	1.26	0.316	2.2	3.0	4.0
100	0.99	4.01	3.99	0.100	63.0	67.0	71.0

GENERAL REFERENCES

Haus, *et al*, "IRE Standards on Methods of Measuring Noise in Linear Twoports, 1959" and "Representation of Noise in Linear Twoports," Proceedings of the IRE, January, 1960, pp 60-74.

Bennett, Electrical Noise, McGraw-Hill, 1960.

Davenport and Root, Random Signals and Noise, McGraw-Hill, 1958.

IRE Subcommittee 7.9 on Noise, "Description of the Noise Performance of Amplifiers and Receiving Systems," Proceedings of the IEEE, March, 1963, pp 436-442.

GENERAL RADIO COMPANY

WEST CONCORD, MASSACHUSETTS, USA

Printed in U.S.A.